5.4 POLYNOMIAL WIGNER-VILLE DISTRIBUTIONS [0]

5.4.1 Polynomial FM Signals

A complex signal can be written in the form

$$z(t) = a(t)\, e^{j\phi(t)}, \tag{5.4.1}$$

where the amplitude $a(t)$ and instantaneous phase $\phi(t)$ are real. Let us define

$$f_z(t) = \frac{\phi'(t)}{2\pi}. \tag{5.4.2}$$

This $f_z(t)$ is the instantaneous frequency (IF) of $z(t)$ *if* $z(t)$ *is analytic*. In this Article we simply use Eqs. (5.4.1) and (5.4.2) as definitions of $z(t)$ and $f_z(t)$, *without* assuming that $z(t)$ is analytic. Results concerning $f_z(t)$ will also be valid for the IF [usually written $f_i(t)$] when $z(t)$ is analytic.

The factor $a(t)$ allows amplitude modulation, phase inversion and time limiting. If $\phi(t)$ is a polynomial function of degree p, so that $f_z(t)$ is a polynomial of degree $p-1$, then $z(t)$ is a **polynomial-phase** or **polynomial FM** signal. If $p > 2$, then $f_z(t)$ is nonlinear, so that $z(t)$ is an example of a **nonlinear FM** signal.

Such nonlinear FM signals occur both in nature and in man-made applications [1]. For example, the sonar systems of some bats use *hyperbolic* and *quadratic* FM signals for echo-location. Some radar systems use *quadratic* FM pulse compression signals. Earthquakes and underground nuclear tests may generate nonlinear FM seismic signals in some long-propagation modes. The altitude and speed of an aircraft may be estimated from the nonlinear IF of the engine noise reaching the ground. Nonlinear FM signals also appear in communications, astronomy, telemetry and other disciplines. As these examples suggest, the problem of estimating the IF of a nonlinear FM signal is of some practical importance.

For a deterministic linear FM signal, the Wigner-Ville distribution (WVD) gives an unbiased estimate of the IF. To obtain the same property with higher-order polynomial FM signals, an extension of the WVD called the **polynomial Wigner-Ville distribution** (**PWVD**) was defined [2,3]. If the instantaneous phase $\phi(t)$ is a polynomial of degree not exceeding p, then the IF estimate based on a PWVD of order p is unbiased [4].

5.4.2 Principles of Formulation of Polynomial WVDs

We seek a Time-Frequency Distribution of the form

$$P_z(t,f) = \underset{\tau \to f}{\mathcal{F}} \{R_z(t,\tau)\} \tag{5.4.3}$$

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where $R_z(t,\tau)$, called the **signal kernel**,[1] somehow depends on $z(t)$. If $a(t) = 1$, then, for the best possible representation of the IF law, we would like $P_z(t,f)$ to be equal to $\delta(f - f_z(t))$. Making this substitution in Eq. (5.4.3) and taking inverse FTs, we find

$$R_z(t,\tau) = e^{j2\pi f_z(t)\tau} = e^{j\phi'(t)\tau}. \tag{5.4.4}$$

So, while $z(t)$ may have a variable frequency with respect to t, we want $R_z(t,\tau)$ to have a constant frequency w.r.t. τ, namely $f_z(t)$. That is, we want the signal kernel to **dechirp** the signal, yielding a sinusoid of constant frequency, for which the FT is the optimal representation and gives a delta function at $f_z(t)$.

To estimate $\phi'(t)$ in Eq. (5.4.4), we use a central finite-difference (CFD) approximation of the form

$$\phi'(t) \approx \widehat{\phi}'(t) = \frac{1}{\tau}\sum_{l=1}^{q} b_l[\phi(t+c_l\tau) - \phi(t-c_l\tau)] \tag{5.4.5}$$

and choose the dimensionless[2] real coefficients b_l and c_l so that the approximation is exact if $\phi(t)$ is a polynomial of degree p. Let that polynomial be

$$\phi(t) = \sum_{i=0}^{p} a_i t^i \tag{5.4.6}$$

so that

$$\phi'(t) = \sum_{i=1}^{p} i a_i t^{i-1}. \tag{5.4.7}$$

A polynomial of degree p remains a polynomial of degree p if the argument is shifted and scaled. If we shift the time origin so that $t = 0$ in Eqs. (5.4.5) to (5.4.7), we see that if $\phi(t)$ contains only even-power terms, both $\phi'(t)$ and $\widehat{\phi}'(t)$ are zero, so that the even-power terms do not introduce any errors into the estimate. So we may assume that p is even and consider only the odd-power terms in $\phi(t)$. There are $p/2$ such terms, hence $p/2$ degrees of freedom in the coefficients a_i, suggesting that the CFD estimate can be made exact by using only $p/2$ finite differences in Eq. (5.4.5), with uniform sampling intervals. But we shall retain the generality of the sampling instants because we can do so without further algebraic complexity, and because the extra degrees of freedom turn out to be useful. With $q = p/2$, Eq. (5.4.5) becomes

$$\phi'(t) = \frac{1}{\tau}\sum_{l=1}^{p/2} b_l[\phi(t + c_l\tau) - \phi(t - c_l\tau)] \tag{5.4.8}$$

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[1] This notation is consistent with the convention that $R_z(t,\tau)$ [Article 2.7] is a generalization of $K_z(t,\tau)$ [Section 2.1.2]. But here, as we shall see, the generalization is in a new direction.

[2] We could allow c_l to have the dimensions of time and dispense with the symbol τ; however, retaining τ will emphasize the correspondence between the PWVD and the ordinary WVD.

where the use of ϕ' instead of $\widehat{\phi'}$ asserts the exactness of the estimate. Substituting this into Eq. (5.4.4), and renaming the signal kernel as $R_z^{(p)}(t,\tau)$ to acknowledge the dependence on p, we obtain

$$R_z^{(p)}(t,\tau) = \prod_{l=1}^{p/2} \left[e^{j\phi(t+c_l\tau)}\, e^{-j\phi(t-c_l\tau)} \right]^{b_l} . \tag{5.4.9}$$

Then, substituting from Eq. (5.4.1) with $a(t) = 1$, we find

$$R_z^{(p)}(t,\tau) = \prod_{l=1}^{p/2} \left[z(t+c_l\tau)\, z^*(t-c_l\tau) \right]^{b_l} . \tag{5.4.10}$$

The resulting TFD, denoted by $W_z^{(p)}(t,f)$ and given by Eq. (5.4.3) as

$$W_z^{(p)}(t,f) = \mathop{\mathcal{F}}_{\tau\to f} \left\{ R_z^{(p)}(t,\tau) \right\} , \tag{5.4.11}$$

is called a **polynomial Wigner distribution** (or **polynomial WD**) of order p.

Thus we arrive at a general definition: *A polynomial WD of order p of the signal $z(t)$ is a function $W_z^{(p)}(t,f)$ given by Eqs. (5.4.10) and (5.4.11), such that the coefficients b_l and c_l satisfy Eq. (5.4.8) when $\phi(t)$ is a polynomial of degree not exceeding p.* In the special case in which $z(t)$ is analytic, the polynomial WD becomes the **polynomial Wigner-Ville distribution** (**PWVD**).

If we put $p=2$, $c_1=\frac{1}{2}$ and $b_1=1$, then $R_z^{(p)}(t,\tau)$ reduces to $z(t+\frac{\tau}{2})\, z^*(t-\frac{\tau}{2})$, which is the instantaneous autocorrelation function (IAF), denoted by $K_z(t,\tau)$. We might therefore describe $R_z^{(p)}(t,\tau)$ as a **polynomial IAF** or **higher-order IAF**.

Eq. (5.4.10) shows that for any $z(t)$, the polynomial IAF is Hermitian in τ. It follows that the polynomial WD is *real*.

5.4.3 IF Estimates with Zero Deterministic Bias

When the IF is a polynomial of degree not exceeding $p-1$, the PWVD of order p gives an *unbiased* estimate of the IF law, as is shown by the following result.

Theorem 5.4.1: *If $z(t)$ and $f_z(t)$ are given by Eqs. (5.4.1) and (5.4.2), where $\phi(t)$ is a polynomial of degree not exceeding p, and if $W_z^{(p)}(t,f)$ satisfies the general definition of a pth-order polynomial WD of $z(t)$, then $W_z^{(p)}(t,f)$ is symmetrical in f about $f=f_z(t)$.*

Proof: Substituting Eq. (5.4.1) into Eq. (5.4.10) and simplifying, we find

$$R_z^{(p)}(t,\tau) = R_a^{(p)}(t,\tau)\, \exp\left(j \sum_{j=1}^{p/2} b_l [\phi(t+c_l\tau) - \phi(t-c_l\tau)] \right) \tag{5.4.12}$$

where $R_a^{(p)}(t,\tau)$, as the notation suggests, is the polynomial IAF for $a(t)$, given by

$$R_a^{(p)}(t,\tau) = \prod_{l=1}^{p/2} \left[a(t+c_l\tau)\, a(t-c_l\tau) \right]^{b_l} . \tag{5.4.13}$$

Because $\phi(t)$ is a polynomial function of degree not exceeding p, and because $W_z^{(p)}(t,f)$ is a pth-order polynomial WD, Eq. (5.4.8) is applicable, so that Eq. (5.4.12) becomes

$$R_z^{(p)}(t,\tau) = R_a^{(p)}(t,\tau)\, e^{j\phi'(t)\,\tau} = R_a^{(p)}(t,\tau)\, e^{j2\pi f_z(t)\,\tau}. \tag{5.4.14}$$

Taking Fourier transforms ($\tau \to f$), we find

$$W_z^{(p)}(t,f) = W_a^{(p)}(t,f) \underset{f}{*} \delta(f - f_z(t)) = W_a^{(p)}(t, f - f_z(t)) \tag{5.4.15}$$

where $W_a^{(p)}(t,f)$ is the corresponding polynomial WD of $a(t)$:

$$W_a^{(p)}(t,f) = \underset{\tau \to f}{\mathcal{F}} \left\{ R_a^{(p)}(t,\tau) \right\}. \tag{5.4.16}$$

From Eq. (5.4.13) we see that $R_a^{(p)}(t,\tau)$ is real and even in τ. Hence, from Eq. (5.4.16), $W_a^{(p)}(t,f)$ is real and even in f. Then, from Eq. (5.4.15), $W_z^{(p)}(t,f)$ is real and symmetrical in f about $f = f_z(t)$. ■

Because a symmetrical distribution is symmetrical about its first moment, Theorem 5.4.1 has the following corollary: *If $z(t)$ has polynomial phase of degree not exceeding p, and if $W_z^{(p)}(t,f)$ is a pth-order polynomial WD of $z(t)$, then the first moment of $W_z^{(p)}(t,f)$ w.r.t. f is equal to $f_z(t)$.*

Being unbiased for higher-degree polynomial FM signals, PWVDs can solve problems that quadratic TFDs cannot [2]. PWVDs also give optimal frequency resolution in the sense that they are FTs of maximal-length polynomial IAFs derived from the full-length signal.

5.4.4 Calculation of Coefficients

Applying Eq. (5.4.6) in Eq. (5.4.8) gives

$$\phi'(t) = \frac{1}{\tau} \sum_{l=1}^{p/2} \left[b_l \sum_{i=0}^{p} a_i \left[(t + c_l\tau)^i - (t - c_l\tau)^i \right] \right]. \tag{5.4.17}$$

Eqs. (5.4.7) and (5.4.17) give two expressions for $\phi'(t)$. Equating these expressions, and shifting and scaling the time variable so that $t = 0$ and $\tau = 1$, we obtain

$$a_1 = \sum_{l=1}^{p/2} \left[b_l \sum_{i=0}^{p} a_i \left[{c_l}^i - (-c_l)^i \right] \right] = 2 \sum_{l=1}^{p/2} \left[\sum_{i=1,3,\ldots}^{p-1} a_i\, {c_l}^i \right] b_l. \tag{5.4.18}$$

In the left-hand and right-hand expressions of this equation, the coefficient of a_i is zero for all even values of i, justifying the decision to consider only odd values of i. Equating coefficients of a_1 gives

$$\frac{1}{2} = \sum_{l=1}^{p/2} c_l b_l\ , \tag{5.4.19}$$

and equating coefficients of a_3 , a_5 ,. . . , a_{p-1} gives

$$0 = \sum_{l=1}^{p/2} {c_l}^i \, b_l \ ; \quad i = 3, 5, \ldots, p-1. \tag{5.4.20}$$

The last two equations can be written in matrix form as

$$\begin{bmatrix} c_1 & c_2 & c_3 & \cdots & c_{p/2} \\ {c_1}^3 & {c_2}^3 & {c_3}^3 & \cdots & {c_{p/2}}^3 \\ {c_1}^5 & {c_2}^5 & {c_3}^5 & \cdots & {c_{p/2}}^5 \\ \vdots & \vdots & \vdots & & \vdots \\ {c_1}^{p-1} & {c_2}^{p-1} & {c_3}^{p-1} & \cdots & {c_{p/2}}^{p-1} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{p/2} \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{5.4.21}$$

and solved algebraically or numerically.

Eq. (5.4.21) is underdetermined: $p/2$ equations in p unknowns leave $p/2$ degrees of freedom. Various ways of exploiting the degrees of freedom give rise to various forms of the PWVD. Here we mention two forms described in [2, p. 217].

If we decide that we want uniform sampling intervals, we choose the values of c_l and accept the resulting values of b_l , which in general turn out to be fractions. The resulting form of the PWVD is called the **fractional-powers form** or **Form I**, of which an example is given in [3, p. 550]. The need to compute fractional powers is a cause of inefficiency.

Alternatively, if we decide that the polynomial IAF must contain only positive integer powers, we choose the values of b_l and accept the resulting values of c_l , which in general give non-uniform sampling intervals. The resulting form of the PWVD is called the **integer-powers form** or **Form II**. In a discrete-time implementation, non-uniform sampling requires interpolation, but this is more efficient than computing non-integer powers. The number of interpolations required can be reduced by time-scaling [2, p. 218].

(There is also a "complex-time form" or "Form III" based on an analytic extension of the signal with a complex time argument; this is described in [5], with further details in [6].)

Notice that given one solution to Eq. (5.4.21), we can obtain another solution by changing the sign of any c_l and of the corresponding b_l. Thus, from one solution, we can always find another solution in which all the indices b_l are positive, ensuring that the polynomial IAF does not become unbounded as the signal approaches zero.

5.4.5 Examples

In the trivial case for which $p = 2$ (quadratic phase, linear FM), Eq. (5.4.21) reduces to $c_1 b_1 = 1/2$. If we fix the sampling points by choosing $c_1 = 1/2$, that leaves $b_1 = 1$. Alternatively, if we choose $b_1 = 1$ for unit powers, that leaves $c_1 = 1/2$. Thus we have a degenerate case in which Forms I and II overlap. Substituting for

p, b_1 and c_1 in Eq. (5.4.8), we obtain the simplest possible CFD approximation:

$$\phi'(t) = \frac{1}{\tau}\left[\phi(t+\tfrac{\tau}{2}) - \phi(t-\tfrac{\tau}{2})\right]. \tag{5.4.22}$$

The same substitutions in Eq. (5.4.10) yield the polynomial IAF

$$R_z^{(2)}(t,\tau) = z(t+\tfrac{\tau}{2})\, z^*(t-\tfrac{\tau}{2}). \tag{5.4.23}$$

This is just the ordinary IAF $K_z(t,\tau)$, which when substituted into Eq. (5.4.11) yields the ordinary (quadratic) Wigner distribution. So, for $p = 2$, Theorem 5.4.1 says that the Wigner distribution is symmetrical about $f_z(t)$ if $z(t)$ is a quadratic-phase (i.e. linear FM) signal. This confirms that the WVD gives an unbiased estimate of the IF for deterministic linear FM signals.

In the case for which $p = 4$ (quartic phase, cubic FM), Eq. (5.4.21) reduces to the 2×2 system

$$\begin{aligned} c_1 b_1 + c_2 b_2 &= 1/2 \\ {c_1}^3 b_1 + {c_2}^3 b_2 &= 0 \end{aligned}\;. \tag{5.4.24}$$

Form I: If we take $c_1 = 1/4$ and $c_2 = -1/2$, Eqs. (5.4.24) become a linear system with solutions $b_1 = 8/3$, $b_2 = 1/3$. Substituting these values into Eq. (5.4.10) gives

$$R_z^{(4)}(t,\tau) = \left[z(t+\tfrac{\tau}{4})\, z^*(t-\tfrac{\tau}{4})\right]^{\frac{8}{3}} \left[z(t-\tfrac{\tau}{2})\, z^*(t+\tfrac{\tau}{2})\right]^{\frac{1}{3}} \tag{5.4.25}$$

where the superscript "(4)" indicates order 4.

Form II: If we take $b_1 = 2$ and $b_2 = 1$, Eqs. (5.4.24) become a nonlinear system with solutions

$$c_1 = \left[2(2-2^{1/3})\right]^{-1} \approx 0.6756 \;\; ; \quad c_2 = -2^{1/3} c_1 \approx -0.8512\;. \tag{5.4.26}$$

Substituting for b_1 and b_2 in Eq. (5.4.10) gives

$$R_z^{(4)}(t,\tau) = \left[z(t+c_1\tau)\, z^*(t-c_1\tau)\right]^2 \; z(t+c_2\tau)\, z^*(t-c_2\tau). \tag{5.4.27}$$

where c_1 and c_2 are as in Eqs. (5.4.26). This is one of an infinite number of Form-II solutions. More details on the design procedure are given in [1,2,7] and in Article 5.5 (next).

Fig. 5.4.1 illustrates the effectiveness of the PWVD in suppressing artifacts caused by errors in the CFD estimate of the IF. The IF of the test signal is

$$f_z(t) = f_\mathrm{c} + f_\mathrm{d}\, C_5(t/T)\;; \quad 0 \le t \le T \tag{5.4.28}$$

where f_c is the center frequency, f_d is the minimax frequency deviation, and C_5 is the Chebyshev polynomial of degree 5. Trace (a) shows the ordinary Wigner-Ville distribution (WVD), which clearly cannot handle the nonlinear IF law. The Form-II PWVD shown in trace (b) was computed by the "*TFSA*" software toolbox, developed in-house by the Signal Processing Research Centre, QUT. It is of 6th order, so that its signal kernel exactly dechirps the 5th-degree IF law. The superiority of the PWVD is evident, as is its symmetry about the IF law.

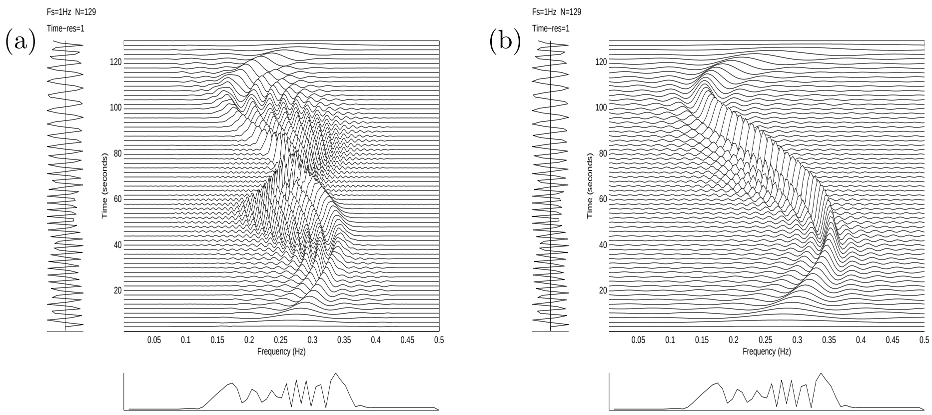

Fig. 5.4.1: Time-frequency representations of a 5th-degree Chebyshev polynomial FM signal [Eq. (5.4.28)] with center frequency $f_c = 0.25\,\mathrm{Hz}$, minimax frequency deviation $f_d = 0.1\,\mathrm{Hz}$, duration $T = 128$ seconds: (a) WVD; (b) 6th-order Form-II PWVD. Both TFDs are unwindowed. Each plot shows time vertically (range 0 to 128 s; resolution 2 s) and frequency horizontally (range 0 to 0.5 Hz), and has the time trace at the left and the magnitude spectrum at the bottom.

5.4.6 Multicomponent Signals and Polynomial TFDs

The use of TFDs for multicomponent signal analysis requires a reliable method of suppressing cross-terms (see Section 3.1.2 and Articles 4.2 and 5.2).

For *quadratic* TFDs, the mechanisms of generation and suppression of cross-terms are well understood. In the WVD, each cross-term is generated midway between the interacting components and alternates at a rate proportional to the separation between the components (see Article 4.2). Other quadratic TFDs may be obtained from the WVD by 2D low-pass filtering, which attenuates the cross-terms because of their alternating (high-pass) character.

For higher-order *polynomial* TFDs, there is the added difficulty that *cross-terms do not necessarily alternate*, so that it may not be possible to suppress cross-terms entirely by convolving them with a simple smoothing function in the (t, f) plane. However, if polynomial TFDs are implemented according to the **S-method**, as described in Article 6.2, the generation of cross-terms can be avoided [8].

5.4.7 Summary and Conclusions

A PWVD of degree p is derived from a p^{th}-order CFD approximation to the derivative of the instantaneous phase. It reduces to the ordinary WVD if $p = 2$. It is real and symmetrical about the IF law of an FM signal whose instantaneous phase is a polynomial of degree not exceeding p (i.e. whose IF is a polynomial of degree not exceeding $p-1$), even if that signal is also amplitude-modulated.

This topic is developed further in Article 5.5 (next). More properties of PWVDs are given in [1]. Some implementation issues are discussed in [4] and [7]. The effect of noise is considered in Article 10.4.

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